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Approximating edge dominating set in dense graphs

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A R T I C L E I N F O

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A B S T R A C T

We study the approximation complexity of the *Minimum Edge Dominating Set* problem in everywhere ϵ-dense and average $\bar{\epsilon}$-dense graphs. More precisely, we consider the computational complexity of approximating a generalization of the Minimum Edge Dominating Set problem, the so called *Minimum Subset Edge Dominating Set* problem. As a direct result, we obtain for the special case of the Minimum Edge Dominating Set problem in everywhere ϵ-dense and average $\bar{\epsilon}$-dense graphs by using the techniques of Karpinski and Zelikovsky, the approximation ratios of $\min\{2, 3/(1+2\epsilon)\}$ and of $\min\{2, 3/(3-2\sqrt{1-\bar{\epsilon}})\}$, respectively.

On the other hand, we give new approximation lower bounds for the Minimum Edge Dominating Set problem in dense graphs. Assuming the Unique Game Conjecture, we show that it is NP-hard to approximate the Minimum Edge Dominating Set problem in everywhere ϵ-dense graphs with a ratio better than $2/(1+\epsilon)$ with $\epsilon > 1/3$ and $2/(2-\sqrt{1-\bar{\epsilon}})$ with $\bar{\epsilon} > 5/9$ in average $\bar{\epsilon}$-dense graphs.

1. Introduction

In this paper, we consider the computational complexity of approximating the *Minimum Subset Edge Dominating Set* problem which generalizes the Minimum Edge Dominating Set problem. As a direct result, we obtain improved upper bounds for the Minimum Edge Dominating Set problem in everywhere and average dense graphs, i.e. graphs with bounded minimum and average vertex degree, respectively.

1.1. Problem statement

An *edge dominating set* (for short EDS) of a finite undirected graph $G = (V, E)$ is a subset $M \subseteq E$ of edges such that each edge in E shares an endpoint with some edges in M. The *Minimum Edge Dominating Set* problem (for short MEDS problem) asks to find an edge dominating set of minimum cardinality $|M|$ (respectively minimum total weight in the weighted case).

For given graph $G = (V, E)$, the *Minimum Maximal Matching* problem (for short MMM problem) asks for a subset $M \subseteq E$ of nonadjacent edges with minimal cardinality such that each edge in E shares an endpoint with some edge in M.

It has been noted long time ago that the Minimum Edge Dominating Set and the Minimum Maximal Matching problem admit optimal solutions of the same size and that an optimal solution of the MEDS problem can be transformed in polynomial time into an optimal solution of the MMM problem (cf. [26]), and vice versa.

The *Minimum Subset Edge Dominating Set* problem (for short MSED problem) is a generalization of the MEDS problem and is defined as follows: given a graph $G = (V, E)$ and a subset $S \subseteq V$, find a minimum cardinality EDS M of G with the property $S \subseteq \bigcup_{e \in M} e$.

For some $\epsilon, \bar{\epsilon} > 0$, we call a graph $G = (V, E)$ *everywhere ϵ-dense* if any vertex in G has at least $\epsilon|V|$ neighbors, and we call a graph $G = (V, E)$ *average $\bar{\epsilon}$-dense* if the average degree of a vertex in G is at least $\bar{\epsilon}|V|$, i.e. $(\sum_{v \in V} deg(v))/|V| \geq \bar{\epsilon}|V|$.

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1.2. Related work

The MEDS problem is already referred to in Garey and Johnson [15]. Even for planar or bipartite graphs of maximum degree 3 the MEDS problem remains *NP*-hard [26] in the exact setting. Some additional hard and polynomial time solvable classes of graphs were given by Horton and Kilakos [18], and much more recently by Demange and Ekim [11]. An inapproximability result was obtained by Chlebík and Chlebíková [9], who showed that it is *NP*-hard to approximate the MEDS problem within any factor better than 7/6. They further showed that the MEDS problem is *NP*-hard to approximate within any constant less than $(7+\epsilon)/(6+2\epsilon)$, in graphs with minimum degree at least ϵn. In the unweighted case, finding an arbitrary maximal matching M provides 2-approximation for the MEDS problem, since each edge in the optimal solution can cover at most two edges of M. The first nontrivial approximation algorithm is due to Gotthilf et al. [16] and achieves an approximation ratio of $2 - c\log(n)/n$, where c is an arbitrary positive constant and n is the number of vertices in the graph. A 21/10-approximation algorithm was given by Car et al. [7] for the *Minimum Weighted Edge Dominating Set* problem, a result which was improved to 2 by Fujito and Nagamochi [13].

Density parameters such as the number of edges $\bar{\epsilon}$ and the minimum degree ϵ have been used in approximation ratios for various optimization problems (see [20] for a detailed survey, [22,19,6,3] for the Vertex Cover problem, and [5,2] for Dominating Set and related problems).

Currently, the best parameterized ratios for the Vertex Cover problem with parameters $\bar{\epsilon}$ and ϵ are $2/(2-\sqrt{1-\bar{\epsilon}})$ and $2/(1+\epsilon)$, respectively [22]. Imamura and Iwama [19] later improved the former result, by generalizing it to depend on both $\bar{\epsilon}$ and $\Delta := \max_{v\in V}\{deg(v)\}$.

As for lower bounds, Clementi and Trevisan [10] as well as Karpinski and Zelikovsky [21] proved that the Vertex Cover problem restricted to everywhere and average dense graphs remains APX-hard. Later, Eremeev [12] showed that it is NP-hard to approximate the Vertex Cover problem in everywhere ϵ-dense graphs within a factor less than $(7+\epsilon)/(6+2\epsilon)$. Finally, Bar-Yehuda et al. [2] prove that if the Vertex Cover problem cannot be approximated within a factor strictly smaller than 2 on arbitrary graphs, then it cannot be approximated within factors smaller than $2/(2-\sqrt{1-\bar{\epsilon}}) - o(1)$ and $2/(1+\epsilon) - o(1)$, respectively, on average and everywhere dense graphs.

For the MEDS problem, Cardinal et al. achieved the first upper bound smaller than 2 for sufficiently dense graphs. More precisely, the obtained approximation ratio is asymptotic to $\min\{2, 1/\epsilon\}$ in everywhere ϵ-dense graphs and to $\min\{2, 1/(1-\sqrt{1-\bar{\epsilon}})\}$ in average $\bar{\epsilon}$-dense graphs [4]. More recently, Cardinal, Langerman, and Levy provided an improved bound on the approximation ratio for the MEDS problem in average dense graphs. This bound is asymptotic to $1/(1-\sqrt{(1-\epsilon)/2})$, which is smaller than 2 when ϵ is greater than 1/2 [5].

1.3. Our contributions

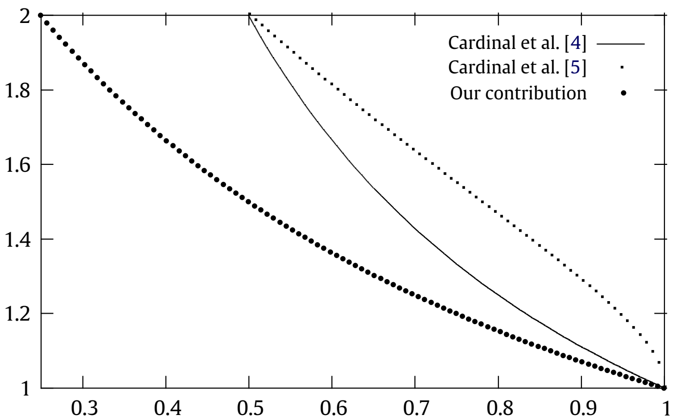

Fig. 1. A comparison of the upper bounds of [4,5], and our contribution for everywhere ϵ-dense graphs. The density parameter ϵ (x-axis) is plotted against the approximation ratio.

This work is the first best to our knowledge studying the approximation complexity of the MSED problem. We give an approximation algorithm that achieves the approximation ratio at most $\min\{2, 3/(1+2|S|/|V|)\}$. For the special case of the MEDS problem in dense graphs, it yields by using the techniques of Karpinski and Zelikovsky for the dense Minimum Vertex Cover problem [22] an approximation ratio of $\min\{2, 3/(1+2\epsilon)\}$ for everywhere ϵ-dense graphs and $\min\{2, 3/(3-2\sqrt{1-\bar{\epsilon}})\}$ for average $\bar{\epsilon}$-dense graphs, respectively. Fig. 1 compares the upper bounds of [4,5], and our contribution for the MEDS problem in everywhere dense graphs. Accordingly, Fig. 2 compares the upper bound of these works for average dense graphs.

On the other hand, we use an approximation preserving reduction due to Karpinski and Zelikovsky [21]) from the Minimum Vertex Cover problem to the Minimum Vertex Cover problem in dense graphs to obtain hardness result for the

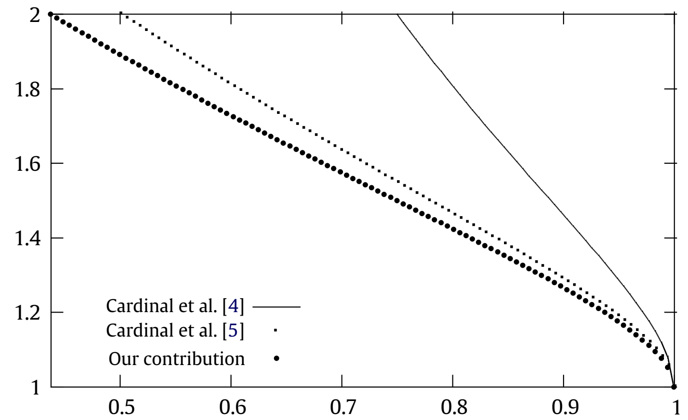

Fig. 2. A comparison of the upper bounds of [4,5], and our contribution for average $\bar{\epsilon}$-dense graphs. The density parameter $\bar{\epsilon}$ (x-axis) is plotted against the approximation ratio.

MEDS problem in dense graphs. Thus assuming the Unique Game Conjecture (cf. [24]), it is NP-hard to approximate the MEDS problem in everywhere ϵ-dense graphs with a ratio better than $2/(1+\epsilon)$ with $\epsilon > 1/3$ and $2/(2-\sqrt{1-\bar{\epsilon}})$ with $\bar{\epsilon} > 5/9$ in average $\bar{\epsilon}$-dense graphs. The same reduction shows that the MSED problem is UGC-hard to approximate within any constant better than $2/(1+|S|/|V|)$ with $3|S| > |V|$.

2. Subset edge dominating set problem

We start by introducing some basic notations and tools which are used in our algorithms. Afterwards we state our approximation algorithm for the MSED problem and prove the claimed result.

2.1. Definitions and notations

Given a finite graph $G = (V, E)$ and a subset $S \subseteq V$, the induced subgraph $G[S]$ is defined as $(S, \{e \in E \mid e \subseteq S\})$. For a given set $M \subseteq E$ we introduce the notation $V(M) := \bigcup_{e \in M} e$.

The maximal matching heuristic is a standard algorithm that provides a 2-approximation for the Minimum Edge Dominating Set problem. It is perhaps one of the simplest and best-known approximation algorithm. It consists in finding a collection of disjoint edges (a matching) that is maximal (with respect to edge inclusion) by iteratively removing adjacent vertices until no more edges are left in the graph.

In the Maximum Subset Matching problem (for short MSM problem), which generalizes the Maximum Matching problem, we are given a graph $G = (V, E)$ and $S \subset V$. The goal is to determine the maximum number of vertices of S that can be matched in a matching of G. Alon and Yuster considered this problem and introduced a randomized algorithm in [1]. The Maximum Subset Matching problem can be reduced to the Maximum Weighted Matching problem. Just assign to every vertex with both endpoints in S weight 2, and edges from S to $V \backslash S$ weight 1. The currently fastest algorithm for maximum weighted matchings in general graphs is the algorithm of Gabow and Tarjan (see [14]).

In our setting, it runs in $\tilde{O}(\sqrt{|V|}(|E| + |S|^2))$ time. For a given graph $G = (V, E)$, $S \subseteq V$ and $U \subseteq V \backslash S$, let us denote by $MSM(G, S, U)$ the set of edges of a maximum subset matching in the graph $G[S \cup U]$ and S.

An important theorem for many problems related to the Minimum Vertex Cover problem was proven by Nemhauser and Trotter (cf. [25]). It enables us to reduce the problem to instances in which the value of a minimum vertex cover is at least $|V|/2$ together with other nice properties. Here, we use a generalized version of the NT-Theorem given by Chlebík and Chlebíková.

Theorem (*Optimal Version of the NT-Theorem [8]*)**.** *There exists a polynomial time algorithm that partitions the vertex set V of any graph G into three subsets $(V_0, V_1, V_{1/2})$ with no edges between V_0 and $V_{1/2}$ or within V_0 such that*

1. *for any vertex cover VC of $G[V_{1/2}]$ it holds $|VC| \geq \frac{|V_{1/2}|}{2}$*
2. *every minimum vertex cover C for G satisfies $V_1 \subseteq C \subseteq V_1 \cup V_{1/2}$ and $C \cap V_{1/2}$ is a minimum vertex cover for $G[V_{1/2}]$.*

Such a partition can be constructed by computing maximum matching of a specially constructed bipartite graph. The algorithm of Hopcroft and Karp is currently the fastest algorithm for maximum matching in bipartite graphs and runs in time $O(|E|\sqrt{|V|})$ (see [17]).

2.2. Algorithm $\mathcal{A}_{SEDS}$

In order to explain the intuition behind the algorithm, notice that the set S needs to be covered with edges and we want to achieve it by a maximum matching which covers the whole set S. Clearly, we cannot expect that there always exists a perfect matching in $G[S]$. Instead we compute a maximum subset matching with endpoints in $V_1 \cup V_{1/2}$ for which we hope to have good vertex cover properties in $G[V \backslash S]$. The remaining vertices of S will be covered greedily. Finally, we take care of the remaining graph by applying the maximal matching heuristic (MMH).

We now present our main algorithm (see Fig. 3).

Input: Graph $G = (V, E)$, $S \subseteq V$

Set $M_1 := \emptyset$;
If $|S| > \frac{|V|}{4}$ **Then**
  Compute the NT-Partition $(V_0, V_1, V_{1/2})$ of $G[V \backslash S]$;
  If $|V_0| < 2|V_1|$ **Then**
    Compute $M_1 := MSM(G, S, V \backslash S)$;
  Else
    Compute $M_1 := MSM(G, S, V_1 \cup V_{1/2})$;
  EndIf
EndIf
Cover the remaining vertices of S greedily with edges M_r;
Compute the remaining graph $G' := G[V \backslash V(M_1 \cup M_r)]$;
Construct a maximal matching M_2 in G' by applying the MMH;

Output: $M_1 \cup M_r \cup M_2$

Fig. 3. Algorithm $\mathcal{A}_{SEDS}$.

2.3. Analysis of $\mathcal{A}_{SEDS}$

We now formulate our main theorem.

Theorem 2.1. *Given a graph $G = (V, E)$ and $S \subseteq V$, the algorithm $\mathcal{A}_{SEDS}$ has an approximation ratio at most* $\min\left\{2, \frac{3}{1+2\frac{|S|}{|V|}}\right\}$.

Proof. Let OPT denote some optimal solution for the MSED problem and $EDS_{\mathcal{A}}$ the solution produced by algorithm $\mathcal{A}_{SEDS}$. First, we concentrate on the case $|S| \le |V|/4$. Then, we show that $\mathcal{A}_{SEDS}$ computes a solution with approximation ratio $3/(1 + 2|S|/|V|)$ which is better than 2 if $|S| > |V|/4$ holds. We start with the following.

Lemma 2.1. *If $|S| \le |V|/4$ holds, then the algorithm $\mathcal{A}_{SEDS}$ has an approximation ratio at most 2.*

Proof. The algorithm covers the vertices of S greedily with edges, which means that we use at most $|S|$ edges. Since the maximal matching heuristic computes a solution as well for the MEDS problem as for the Minimum Vertex Cover problem (by choosing the endpoints of the constructed matching) with approximation ratio 2, our solution for the graph $G[V \backslash S]$ has at most as many edges as the cardinality of an optimal vertex cover VC_{OPT} of $G[V \backslash S]$. Consequently, the approximation ratio of the algorithm is bounded by

$$\frac{|EDS_{\mathcal{A}}|}{|OPT|} \le \frac{|S| + |VC_{OPT}|}{\frac{1}{2}(|S| + |VC_{OPT}|)} = 2. \quad \square$$

In the remaining part of the proof, we will restrict ourselves to instances (G, S) with $|S| > |V|/4$.

For the sake of the analysis, let us now consider a maximum subset matching $M^* := MSM(G^*, S, V(OPT) \cap V')$ of the restricted graph $G^* = (V(OPT), OPT)$, where V' is a subset of $V \backslash S$. We denote by M_R^* the edges contained in OPT to cover the remaining vertices in $S \backslash V(M^*)$, i.e. $M_R^* := \{e \in OPT \mid e \cap (S \backslash V(M^*)) \neq \emptyset\}$. We prove a simple lemma.

Lemma 2.2. *Let M be a maximal subset matching $MSM(G, S, V')$ and $M_r \subseteq E(G)$ be the edges which are greedily chosen to cover the remaining vertices in $S \backslash V(M)$. Then we have $|M_r| \le |M_R^*|$.*

Proof. Since OPT is contained in $E(G)$ and by definition of a maximal subset matching, it is clear that $|S \cap V(M^*)| \le |S \cap V(M)|$ holds. Therefore, we conclude $|S \backslash V(M^*)| \ge |S \backslash V(M)|$ which implies $|M_r| \le |M_R^*|$. $\square$

Let us assume that $|S| > |V|/4$ holds, we now show that $\mathcal{A}_{SEDS}$ has an approximation ratio at most $3/(1 + |S|/|V|)$. We will consider two cases separately.

Case $|V_0| < 2|V_1|$**:**

First of all, the algorithm $\mathcal{A}_{SEDS}$ computes a maximum subset matching $M_1 := MSM(G, S, V \backslash S)$ of G and then covers the remaining vertices of S greedily with edges M_r (see Fig. 4).

Let $M^* := MSM(G^*, S, V(OPT) \backslash S)$ be a maximum subset matching of the restricted graph $G^* = (V(OPT), OPT)$ and denote by M_R^* the edges contained in OPT to cover the vertices in $S \backslash V(M^*)$. From Lemma 2.2 we know that $|M_r| \le |M_R^*|$ holds.

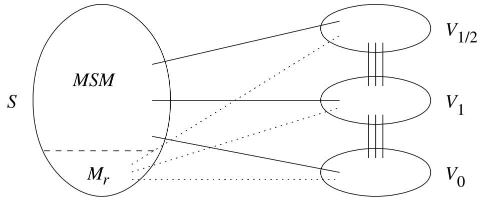

Fig. 4. The partition of G in the case of $|V_0| < 2|V_1|$.

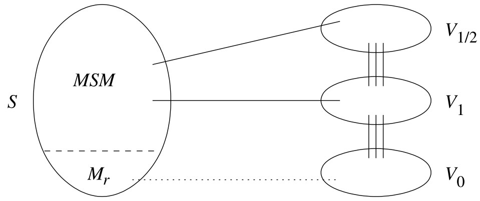

Fig. 5. The partition of G in the case of $|V_0| \geq 2|V_1|$.

We analyze the cardinality of $EDS_{\mathcal{A}}$, the solution produced by $\mathcal{A}_{SEDS}$, and OPT separately. The maximum subset matching $MSM(G, S, V \backslash S)$ covers in the worst case all the vertices of the remaining graph $G[V \backslash S]$ and $|S| - |M_r|$ vertices of S. Therefore, we can bound the cardinality of $EDS_{\mathcal{A}}$ as follows:

$$2|EDS_{\mathcal{A}}| \leq (|S| - |M_r|) + |V \backslash S| + 2|M_r| \leq |V| + |M_r| \leq |V| + |M_R^*|.$$

Now we give a lower bound on the optimal solution. Notice that the cardinality of $V(OPT) \backslash S$ is at least $|V_1| + \frac{1}{2}|V_{1/2}|$, since $|V_1| + |V_{1/2}|/2$ is a lower bound on the cardinality of an optimal vertex cover of $G[V \backslash S]$. Therefore, we can assume that a matching in OPT covers the $|V_1| + \frac{1}{2}|V_{1/2}|$ vertices in $G[V \backslash S]$ and $|S| - |M_R^*|$ vertices in S. The remaining vertices in S are covered by $|M_R^*|$ edges. Hence, we get the following:

$$\begin{aligned} 2|OPT| &\geq (|S| - |M_R^*|) + |V_1| + \frac{1}{2}|V_{1/2}| + 2|M_R^*| \\ &\geq |S| + |V_1| + \frac{1}{2}|V_{1/2}| + |M_R^*|. \end{aligned}$$

We are ready to analyze the approximation ratio of $\mathcal{A}_{\text{SEDS}}$ by combining the upper and lower bounds. In (1), we use the property of the case $|V_0| < 2|V_1|$.

$$\begin{aligned} \frac{2|EDS_{\mathcal{A}}|}{2|OPT|} &\leq \frac{|V| + |M_R^*|}{|S| + |V_1| + \frac{1}{2}|V_{1/2}| + |M_R^*|} \leq \frac{|V|}{|S| + |V_1| + \frac{1}{2}|V_{1/2}|} \\ &\leq \frac{3}{\frac{3|S|+3|V_1|+\frac{3}{2}|V_{1/2}|}{|V|}} \leq \frac{3}{\frac{|S|+3|V_1|+|V_{1/2}|}{|S|+|V_1|+|V_{1/2}|+|V_0|} + \frac{2|S|+\frac{1}{2}|V_{1/2}|}{|V|}} \\ &\leq \frac{3}{\frac{|S|+3|V_1|+|V_{1/2}|}{|S|+3|V_1|+|V_{1/2}|} + \frac{2|S|+\frac{1}{2}|V_{1/2}|}{|V|}} \\ &\leq \frac{3}{1 + \frac{2|S|+\frac{1}{2}|V_{1/2}|}{|V|}} \leq \frac{3}{1 + 2\frac{|S|}{|V|}} \end{aligned} \tag{1}$$

Case $|V_0| \geq 2|V_1|$:
Unlike the previous case, the algorithm $\mathcal{A}_{\text{SEDS}}$ computes a maximum subset matching $MSM(G, S, V_1 \cup V_{1/2})$ of G (see Fig. 5). As before M_r and M_R^* are the sets of edges to cover the remaining vertices of S, where $V(M_R^*) \cap S$ are the vertices left uncovered by a maximum subset matching $M^* := MSM(G^*, S, (V_1 \cup V_{1/2}) \cap V(OPT))$ of $G^* := (V(OPT), OPT)$. From Lemma 2.2 we know that $|M_r| \leq |M_R^*|$ holds.

As before, we analyze $EDS_{\mathcal{A}}$ and OPT separately. This time the algorithm $\mathcal{A}_{SEDS}$ computes a maximum subset matching $MSM(G, S, V_1 \cup V_{1/2})$ which contains in the worst case only the vertices in $S \backslash V(M_r)$. Afterwards, the Maximal Matching Heuristic produces a matching which covers $2|V_1| + |V_{1/2}|$ vertices of the remaining graph $G[V \backslash S]$. In this way, we derive the following:

$$\begin{aligned} 2|EDS_{\mathcal{A}}| &\leq (|S| - |M_r|) + 2|V_1| + |V_{1/2}| + 2|M_r| \\ &\leq |S| + 2|V_1| + |V_{1/2}| + |M_r| \\ &\leq |S| + 2|V_1| + |V_{1/2}| + |M_R^*|. \end{aligned}$$

Now we analyze the cardinality of OPT. In contrast to the previous case, the independent set $G[V_0]$ is sufficiently large. Some of the vertices of $V(M_R^*) \cap V_0$ could be used to cover edges between V_0 and V_1. Nevertheless, the number of such edges is bounded by $|V_1|$, since $|V_1| + |V_{1/2}|/2$ is a lower bound on the cardinality of an optimal vertex cover of $G[V \setminus S]$. The crucial fact $|M_R^*| \geq |V_1|$ will be used later on to attain (2). We give a lower bound on the cardinality of OPT.

$$\begin{aligned} 2|OPT| &\geq (|S| - |M_R^*|) + \frac{1}{2}|V_{1/2}| + 2|M_R^*| \\ &= |S| + \frac{1}{2}|V_{1/2}| + |M_R^*|. \end{aligned}$$

By combining the deduced upper and lower bounds, we analyze the approximation ratio of $\mathcal{A}_{SEDS}$.

$$\begin{aligned} \frac{2|EDS_{\mathcal{A}}|}{2|OPT|} &\leq \frac{|S| + 2|V_1| + |V_{1/2}| + |M_R^*|}{|S| + \frac{1}{2}|V_{1/2}| + |M_R^*|} \\ &\leq \frac{|S| + 2|V_1| + |V_{1/2}| + |V_1|}{|S| + \frac{1}{2}|V_{1/2}| + |V_1|} \\ &\leq \frac{3}{\frac{3(|S|+|V_1|+\frac{1}{2}|V_{1/2}|)}{|S|+3|V_1|+|V_{1/2}|}} \\ &\leq \frac{3}{\frac{|S|+3|V_1|+|V_{1/2}|}{|S|+3|V_1|+|V_{1/2}|} + \frac{2|S|+\frac{1}{2}|V_{1/2}|}{|S|+3|V_1|+|V_{1/2}|}} \\ &\leq \frac{3}{1 + \frac{2|S|+\frac{1}{2}|V_{1/2}|}{|V|}} \leq \frac{3}{1 + 2\frac{|S|}{|V|}}. \quad \square \end{aligned} \tag{2}$$

3. Dense instances of the MEDS problem

In this section, we consider the Minimum Edge Dominating Set problem in dense graphs. First, we start with an observation of fundamental importance to our analysis.

Observation 3.1. *Given a connected graph $G = (V, E)$ and an optimal EDS M of G. There is a vertex $v \in V$ with $N(v) \subseteq V(M)$.*

Proof. If M covers the whole vertex set V, then we have nothing to show. Otherwise the whole neighborhood of a vertex $v \in V \setminus V(M)$ belongs to $V(M)$ to cover the edges incident to v. $\square$

This observation gives us a simple proof of the analysis of the approximation ratio of the maximal matching heuristic in dense graphs studied by Cardinal et al. (see [4]). Since the cardinality of an optimal EDS of an everywhere ϵ-dense graph $G = (V, E)$ can be lower bounded by $\min_{v \in V}\{|N(v)|\}/2 \geq \epsilon|V|/2$ and the worst case solution of the maximal matching heuristic is a maximum matching, the approximation ratio is bounded by $\min\{2, (|V|/2)/(\epsilon|V|/2)\}$.

Next, we want to derive an equivalent statement for average $\bar{\epsilon}$-dense graphs. We need a lemma which was proven by Karpinski and Zelikovsky.

Lemma 3.1 ([22]). *Given an $\bar{\epsilon}$-average dense graph $G = (V, E)$ and let W be the set of $(1 - \sqrt{1 - \bar{\epsilon}})|V|$ vertices with highest degree. Then every vertex of W has degree at least $|W|$.*

As a direct consequence, we get the following.

Corollary 3.1. *Given an $\bar{\epsilon}$-average dense graph $G = (V, E)$. The cardinality of an optimal EDS M is at least $(1 - \sqrt{1 - \bar{\epsilon}})\frac{|V|}{2}$.*

Proof. If the whole set W of $(1 - \sqrt{1 - \bar{\epsilon}})|V|$ vertices with highest degree belongs to $V(M)$, we have nothing to show. Otherwise the neighborhood of a vertex $v \in W \setminus V(M)$ is a subset of $V(M)$. According to Lemma 3.1 the degree of this vertex v is at least $(1 - \sqrt{1 - \bar{\epsilon}})|V|$. Therefore, the cardinality of M can be lower bounded by $|N(v)|/2 \geq (1 - \sqrt{1 - \bar{\epsilon}})|V|/2$. $\square$

Analogously, one can easily deduce similarly to Observation 3.1 that the maximal matching heuristic computes an EDS in average $\bar{\epsilon}$-dense graphs with approximation ratio at most $\min\{2, (1 - \sqrt{1 - \bar{\epsilon}})^{-1}\}$ as analyzed in [4].

We are ready to state the algorithm for the dense MEDS problem (see Fig. 6).

Corollary 3.2. *The algorithm $\mathcal{A}_{DEDS}$ has an approximation ratio at most* $\min\{2, \frac{3}{1+2\epsilon}\}$ *for ϵ-everywhere dense graphs and at most* $\min\{2, \frac{3}{3-2\sqrt{1-\bar{\epsilon}}}\}$ *for $\bar{\epsilon}$-average dense graphs. $\mathcal{A}_{DEDS}$ has a better approximation ratio than 2 if $\epsilon > 1/4$ or $\bar{\epsilon} > \frac{7}{16}$.*

Proof. Given an ϵ-everywhere dense graph $G = (V, E)$ and an optimal EDS M, $V(M)$ contains always the neighborhood $N(v)$ of a vertex $v \in V$ because of Observation 3.1. By exhaustive search we find the right vertex v and use the algorithm $\mathcal{A}_{SEDS}$ for the MSED problem. In case of $\epsilon \leq 1/4$, we know from Theorem 2.1 that $\mathcal{A}_{SEDS}$ produces a solution with approximation

Input: Graph $G = (V, E)$

ForAll $v \in V$
  compute $\mathcal{A}_{\text{SEDS}}(G, N(v))$;
EndForAll
Let M_1 be the solution with smallest cardinality among $\{\mathcal{A}_{\text{SEDS}}(G, N(v)) \mid v \in V\}$;
Let W be the set of $(1 - \sqrt{1 - \bar{\epsilon}})|V|$ vertices with highest degree;
Compute $M_2 := \mathcal{A}_{\text{SEDS}}(G, W)$;
ForAll $v \in W$
  compute $\mathcal{A}_{\text{SEDS}}(G, N(v))$;
EndForAll
Let M_3 be the solution with smallest cardinality among $\{\mathcal{A}_{\text{SEDS}}(G, N(v)) \mid v \in W\}$;

Output: The best solution among M_1, M_2 and M_3

Fig. 6. Algorithm $\mathcal{A}_{\text{DEDS}}$.

ratio at most 2. Restricted to ϵ-everywhere dense graphs with $\epsilon > 1/4$, we get a solution with an approximation ratio at most

$$\frac{3}{1 + 2\frac{|N(v)|}{|V|}} \le \frac{3}{1 + 2\frac{\epsilon|V|}{|V|}}.$$

In the case of $\bar{\epsilon}$-average dense graphs, we have to consider two cases. If there is a vertex $v \in W$, which does not belong to $V(M)$, then we use the same argumentation as before. Since the smallest degree of a vertex in W is at least $(1 - \sqrt{1 - \bar{\epsilon}})|V|$, the approximation ratio can be bounded as follows:

$$\frac{3}{1 + 2\frac{|N(v)|}{|V|}} \le \frac{3}{1 + 2(1 - \sqrt{1 - \bar{\epsilon}})} = \frac{3}{3 - 2\sqrt{1 - \bar{\epsilon}}}.$$

Otherwise the whole set W belongs to $V(M)$. Since the cardinality of W is $(1 - \sqrt{1 - \bar{\epsilon}})|V|$, the corollary follows from Theorem 2.1. □

4. Approximation hardness results

Assuming the Unique Game Conjecture (see [23]), we provide new lower bounds on efficient approximability for everywhere ϵ-dense (resp. average $\bar{\epsilon}$-dense) instances of the MEDS problem with $1/3 < \epsilon$ (resp. with $5/9 < \bar{\epsilon}$). The starting point of our proof is the hardness result of Khot and Regev [24]. Then we show that the approximation preserving reduction from the Minimum Vertex Cover problem to the dense Vertex Cover problem due to Karpinski and Zelikovsky [21] can be used to derive the claimed inapproximability result for the dense MEDS problem.

We now formulate our inapproximability result.

Theorem 4.1. *For every $\delta > 0$, it is unique game conjecture hard to approximate the everywhere ϵ-dense MEDS problem for every constant $\epsilon, \bar{\epsilon}$ with $\epsilon > \frac{1}{3}$ (resp. average $\bar{\epsilon}$-dense MEDS problem with $\bar{\epsilon} > \frac{5}{9}$) to within $\frac{2}{1+\epsilon} - \delta$ (resp. $\frac{2}{2-\sqrt{1-\bar{\epsilon}}} - \delta$).*

Proof. Khot and Regev [24] showed that for every $\delta > 0$ there are instances $G = (V, E)$ of the Vertex Cover problem such that it is UGC-hard to decide whether $|OPT_{VC}| > (1 - \delta)|V|$ or $|OPT_{VC}| \le (1/2 + \delta)|VC|$. We set $\delta \in (0, \epsilon/(1 - \epsilon) - 1/2)$. Given such an instance, we densify it by joining all vertices of a clique of size $\epsilon/(1 - \epsilon)|V|$ with all vertices of G. The same reduction was used by Karpinski and Zelikovsky ([21]) to prove that the dense Vertex Cover problem is APX-hard. This new instance G' is ϵ-dense, since every vertex of G' has a vertex degree at least

$$\frac{\epsilon}{1 - \epsilon} \cdot n = \frac{\epsilon}{1 - \epsilon} \cdot \frac{n'}{1 + \frac{\epsilon}{1-\epsilon}} = \frac{\epsilon}{1 - \epsilon} \cdot \frac{n'}{\frac{1-\epsilon}{1-\epsilon} + \frac{\epsilon}{1-\epsilon}} = \epsilon \cdot n'.$$

If the optimal solution of the Vertex Cover problem $\le (1/2 + \delta)|V|$, then we can match every vertex in the optimal solution with some vertices in the clique K which is of size $\epsilon n/(1 - \epsilon) > (1/2 + \delta)n$. Since K is a clique, every remaining vertex in K can be matched by edges in $E(K)$ (we can double the graph G and join it with a twice larger clique K' to obtain a perfect matching in $G'[OPT_{VC} \cup V(K')]$). Therefore, the optimal solution for dense MEDS problem is $\le n/2(1/2 + \epsilon/(1 - \epsilon) + \delta)$. If the optimal solution of the Vertex Cover problem is larger than $n(1 - \delta)$, we know that the optimal solution of the dense MEDS problem must be at least $n/2(1 + \epsilon/(1 - \epsilon) - \delta)$, since $V(OPT_{EDS})$ is a vertex cover of the graph G'.

Hence, we get the following UGC-hard decision question:

$$\frac{OPT_{EDS}}{n} \ge \frac{1}{2} + \frac{1}{2}\frac{\epsilon}{1 - \epsilon} - \frac{\delta}{2} \quad \text{or} \quad \frac{OPT_{EDS}}{n} \le \frac{1}{4} + \frac{1}{2}\frac{\epsilon}{1 - \epsilon} + \frac{\delta}{2}.$$

This decision question implies directly the following inapproximability factor:

$$\left(\frac{1}{2}+\frac{1}{2}\frac{\epsilon}{1-\epsilon}-\frac{\delta}{2}\right)\left(\frac{1}{4}+\frac{1}{2}\frac{\epsilon}{1-\epsilon}+\frac{\delta}{2}\right)^{-1} \leq \frac{1-\epsilon+\epsilon}{2(1-\epsilon)}\cdot\frac{4(1-\epsilon)}{1-\epsilon+2\epsilon}-\delta' \leq \frac{2}{1+\epsilon}-\delta'. \tag{3}$$

In the case of average $\bar{\epsilon}$-dense instances of the Minimum Edge Dominating Set problem, we set $\epsilon := 1-\sqrt{1-\bar{\epsilon}}$ and the claimed inapproximability factor follows from (3). It remains to verify that the resulting graph G' is $\bar{\epsilon}$-dense:

$$\sum_{v\in V'}\frac{deg(v)}{|V'|^2} \geq \frac{\epsilon|V'|\overbrace{(|V'|)}^{deg}+n\overbrace{(\epsilon|V'|)}^{deg}}{|V'|^2} = (1-\epsilon)\epsilon+\epsilon = \epsilon(1-\epsilon+1)$$
$$= (1-\sqrt{1-\bar{\epsilon}})(1+\sqrt{1-\bar{\epsilon}}) = 1-1+\bar{\epsilon}. \quad \square$$

Using the same reduction for the MSED problem with $S = V(K)$, we get the following.

Corollary 4.1. *For every $\delta > 0$ and $3|S| \geq |V|$, it is UGC-hard to approximate the MSED problem within $\frac{2}{1+\frac{|S|}{|V|}} - \delta$.*

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